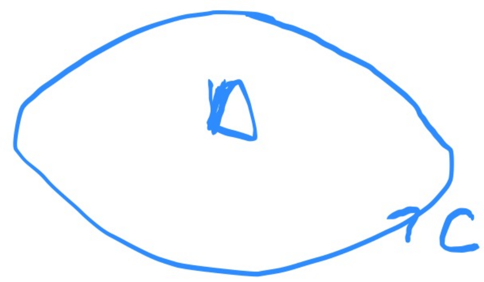
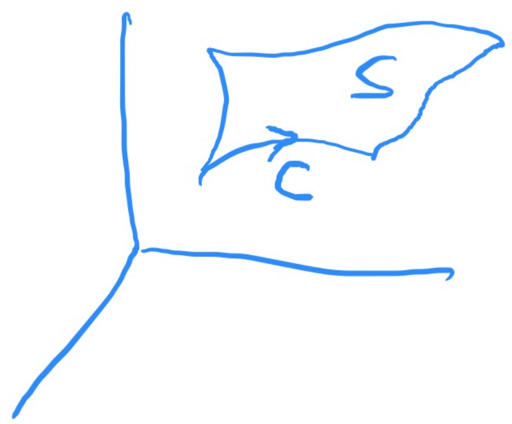
Green's Theorem:

relates a double integral over region $D \subset \mathbb{R}^2$
to a line integral over its boundary C



Stokes' Theorem is generalization of Green's Theorem to surfaces $S \subset \mathbb{R}^3$
with boundary curve C



Recall: If $F: \mathbb{R}^3 \to \mathbb{R}^3$ vector field

$$\Rightarrow \operatorname{curl} F = \nabla \times F \qquad \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

Stokes' Theorem for graphs

Let S be graph of function $f: D \to \mathbb{R}^3$

S given by points (x,y,z) where $z = f(x,y)$ and (x,y) in D

C boundary curve of S

Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field

$$\Rightarrow \boxed{\iint_S \operatorname{curl} F \cdot dS = \int_C F \cdot ds}$$

⚠ integrals of vector fields depend on orientation! theorem only valid for compatible orientation.

- for surface integrals we need to specify which side is the positive side.

  here: positive side = upper side.

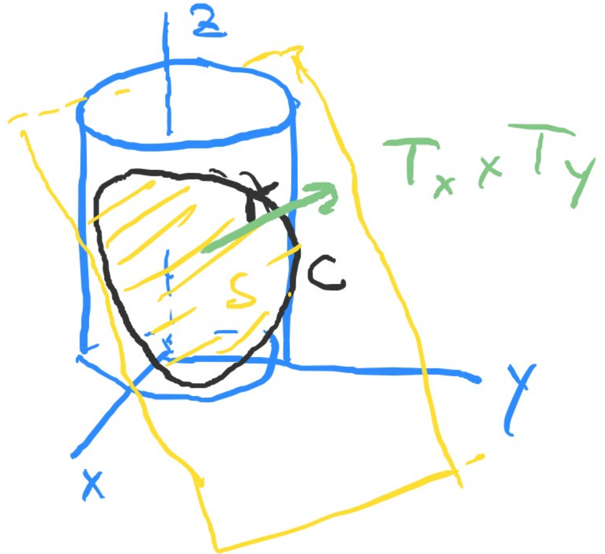
- for line integrals: need to specify direction in which we run through curve.

<u>here:</u> need to walk on positive side of S on C such that S is to our left.

<u>Example:</u> Use Stokes' Theorem to calculate

$$\int_C F \cdot ds \qquad \text{where}$$

C is the intersection of $2x+2y+z=2$ with cylinder $x^2+y^2=1$ oriented counterclockwise around z-axis.



$$\boxed{F(x,y,z) = (-y^3, x^3, -z^3)}$$

Solution: We apply Stokes' Theorem to the part of the plane $2x+2y+z=2$ which is inside the cylinder

Parametrize S: Solve for z : $\boxed{\begin{array}{l} z = 2-2x-2y \\ x^2+y^2 \leq 1 \end{array}}$

Recall: normal vector for graph $z = g(x,y)$ given by $\boxed{T_x \times T_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)}$ (or calculate directly)

here: $g(x,y) = 2-2x-2y \Rightarrow \frac{\partial g}{\partial x} = -2, \frac{\partial g}{\partial y} = -2 \Rightarrow \boxed{T_x \times T_y = (2,2,1)}$

have checked: orientations are compatible

(walking around counterclockwise compatible with normal vector pointing upwards (i.e. its z-coordinate is positive))

Stokes Theorem $\Rightarrow$

$$\int_C F \cdot ds = \iint_S \operatorname{curl} F \cdot dS$$

S parametrized by $\Phi(u,v) = (u, v, 2-2u-2v)$

$$= \iint_{u^2+v^2 \leq 1} \operatorname{curl} F\,(u, v, 2-2u-2v) \cdot (2,2,1)\, du\, dv =$$

$F(x,y,z) = (-y^3, x^3, -z^3)$

$$\operatorname{curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & -z^3 \end{vmatrix} = \vec{i}\,(0) - \vec{j}\,0 + \vec{k}\,(3x^2 + 3y^2)$$

$$= \iint_{u^2+v^2\le 1} \underbrace{(0,0,3u^2+3v^2)\cdot(2,2,1)}_{\text{curl} F \cdot (\Phi(u,v))}\, du\, dv$$

$$= \iint_{u^2+v^2\le 1} 3u^2+3v^2 \, du\, dv$$

$$= \int_0^{2\pi}\int_0^1 3r^2 \cdot r \, dr\, d\theta = \text{easy}$$

$$= \frac{3\pi}{2}$$

polar coordinates

$u = r\cos\theta$

$v = r\sin\theta$

Stokes' Theorem for general case:

$S \subset \mathbb{R}^3$ surface parametrized by

$$\Phi : D \to \mathbb{R}^3$$

S has boundary curve C

$F: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

$$\Rightarrow \int_C F \cdot ds = \iint_S \operatorname{curl} F \cdot dS$$

≈ tangent vector

≈ normal vector

if orientations are compatible

i.e. if we walk on C on the side of normal vector $T_u \times T_v$

→ need to walk in direction s.t. S is to our left

→ determines orientation for C

e.g.